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A note on MHD channel flow under alternating pressure gradient

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The method of Laplace Transform has been employed here to investigate the flow of a conducting liquid between a pair of parallel plates. The liquid is subjected to alternating pressure gradient along its direction of flow and a uniform transverse magnetic field Criteria for laminar and turbulant motions is found to rest on two parameters, the Hartmann number and the Reynolds number appearing in the expression for velocity Motion is found to be purely laminar for large Hartmann number and low Reynolds number, whereas turbulance appears for smaller Hartmann number and higher Reynolds number.

INTRODUCTION

Magnetohydrodynamics is one of the most widely investigated subjects now-a-days. Briefly, it is a combined study of the electromagnetic and hydrodynamic equations. These are systems of non-linear differential equations and it is extremely difficult to achieve their solutions in the general form. One of the first steps to understand magnetohydrodynamics is to seek some simple exact solutions of these equations that reveal the essential features of the problems.

In the present paper we have attempted to derive the expressions for the unsteady flow of a conducting liquid between two parallel plates under alternating pressure gradient and a constant transverse magnetic field, following the method of Laplace Transform (Churchill 1958) Expressions for velocities both due to the presence and absence of the magnetic field are calculated and found to be predominantly transient in nature. The roles of the two important parameters, the Hartmann number and Reynolds number, present in the transient part, in distinguishing between laminar and turbulent motions have been critically surveyed. It has been observed that for a strong field and small Reynolds number motion is purely laminar, whereas, motion is turbulent for a weak field and large Reynolds number (Dube 1969).

EXPLANATION OF THE SYMBOLS USED

H = magnetic field.

σ = conductivity of the liquid.

μ = permeability of the media.

t = variable time.

V = velocity vector of any points of the liquid.

ρ = density of the liquid.

p = pressure of any point of the liquid.

ν = kinematic viscosity of the liquid.

x = variable along the direction of flow.

y = variable perpendicular to direction of flow

PROBLEM, ITS BOUNDARY CONDITIONS AND SOLUTION

Flow problems of electrically conducting liquid, where electromotive forces are of the same order of magnitude as pressuere and viscous forces, are generally met by electromagnetic and hydrodynamic equations known as MHD equations (Clobe 1959) These are systems of non-linear differential equations extremely difficult to solve in their general form. Consequently some simplifications are necessary to achieve their solutions. In the present case we suppose,

$$V = (u, 0, 0), \qquad H = (0, H_y, 0), \quad \frac{\partial}{\partial z}(\quad) = 0.$$

If we further assume that the motion is set up due to alternating pressure gradient of period $2\pi/\omega$ the MHD equations reduce to a single equation of motion

$$\nu\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial t} = -A\cos\omega t + \frac{\mu^2\sigma}{\rho} H^2{}_y u \qquad \dots \quad (1.0)$$

where A is a constant

If $\bar{u} = \int\limits_0^\infty \exp(-St)u dt$, be the Laplace transform of u with initial value u_0, multiplying (1.0) by $\exp(-St)$ and then integrating between 0 and ∞,

$$\frac{\partial^2\bar{u}}{\partial y^2} - \frac{S}{\nu}\bar{u} = -\frac{u_0}{\nu} - \frac{A}{\nu}\frac{S}{S^2+\omega^2} + \frac{\mu^2\sigma}{\rho\nu} H^2{}_y\bar{u} \qquad \dots \quad (2.0)$$

The value of u_0 as given by (1 0) is,

$$\nu\frac{\partial^2 u_0}{\partial y^2} = -A + \frac{\mu^2\sigma}{\rho} H^2{}_y u_0,$$

i.e.,

$$u_0 = A_1\cosh py + B_1\sinh py + \frac{A}{\nu p^2} \qquad \dots \quad (3.0)$$

where,

$$p^2 = \frac{\mu^2\sigma}{\nu\rho} H^2{}_y, \qquad \dots \quad (3\ 1)$$

and A_1 and B_1 are constants eliminated by conditions :

$$\left.\begin{aligned} &\text{at } y=0, \quad u_0=0, \\ \text{and} \quad &\text{at } y=a, \quad u_0=0. \end{aligned}\right\} \quad \ldots \quad (4.0)$$

The value of u_0 as obtained with the help of (3.0) and (4.0) is,

$$u_0 = \frac{A}{\nu p^2}\left[1-\frac{\cosh p(y-a/2)}{\cosh p(a/2)}\right] \quad \ldots \quad (5.0)$$

Therefore the solution of the equation (2 0) can be written as,

$$\bar{u} = A' \cosh\left(\left(\frac{S+\nu p^2}{\nu}\right)^{\frac{1}{2}} y\right) + B' \sinh\left(\left(\frac{S+\nu p^2}{\nu}\right)^{\frac{1}{2}} y\right)$$

$$+\frac{A}{\nu p^2}\left[\frac{1}{S+\nu p^2}-\frac{1}{S}\frac{\cosh p(y-a/2)}{\cosh p(a/2)}+\frac{S\nu p^2}{(S+\nu p^2)(S^2+\omega^2)}\right] \quad \ldots \quad (6.0)$$

Here A' and B' are constants being determined by the conditions :

$$\left.\begin{aligned} &\text{at } y=0, \quad \bar{u}=0, \\ \text{and} \quad &\text{at } y=a, \quad \bar{u}=0. \end{aligned}\right\} \quad \ldots \quad (7.0)$$

Hence,

$$\bar{u} = \frac{A}{\nu p^2\Omega}\left[\frac{\omega}{S+\nu p^2}+\frac{\nu p^2(\omega^2+S\nu p^2)}{S^2+\omega^2}\right]\left\{1-\frac{\cosh\left\{\left(\frac{S+\nu p^2}{\nu}\right)^{\frac{1}{2}}(y-a/2)\right\}}{\cosh\left\{\left(\frac{S+\nu p^2}{\nu}\right)^{\frac{1}{2}}a/2)\right\}}\right\}$$

$$-\frac{A}{\nu p^2\Omega S}\left[\frac{\cosh p(y-a/2)}{\cosh p(a/2)}-\frac{\cosh\left\{\left(\frac{S+\nu p^2}{\nu}\right)^{\frac{1}{2}}(y-a/2)\right\}}{\cosh\left\{\left(\frac{S+\nu p^2}{\nu}\right)^{\frac{1}{2}}a/2\right\}}\right], \quad \ldots \quad (8.0)$$

where, $\Omega = \omega^2+\nu^2p^4$.

To evaluate u we shall need to evaluate the following inversion :

$$\bar{u} = \frac{A}{\nu p^2\Omega}\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}\exp(St)\left[\frac{\omega^2}{S+\nu p^2}+\frac{\nu p^2(\omega^2+S\nu p^2)}{S^2+\omega^2}-\frac{\Omega}{S}\frac{\cosh p(y-a/2)}{\cosh p(a/2)}\right]dS$$

$$-\frac{A}{\nu p^2\Omega}\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}\exp(St)\left[\frac{\omega^2}{S+\nu p^2}+\frac{\nu p^2(\omega^2+S\nu p^2)}{S^2+\omega^2}-\frac{\Omega}{S}\right]$$

$$\frac{\cosh\left\{\left(\frac{S+\nu p^2}{\nu}\right)^{\frac{1}{2}}(y-a/2)\right\}}{\cosh\left\{\left(\frac{S+\nu p^2}{\nu}\right)^{\frac{1}{2}}a/2\right\}}dS \quad \ldots \quad (9.0)$$

The singular points of the above functions are the simple poles at $S = 0$, $-\nu p^2$, $\pm i\omega$ and the infinite number of poles,

$$S = -p^2\nu - \frac{(2n-1)^2\pi^2\nu}{a^2}, \qquad \dots (10.0)$$

obtained by solving,

$$\cosh\left\{\left(\frac{S+\nu p^2}{\nu}\right)^{\frac{1}{2}} \frac{a}{2}\right\} = 0 \qquad \dots (10.1)$$

Now calculating the residues and applying the well-known Cauchy's Residue Theorem,

$$u = \frac{A}{\Omega^{\frac{1}{2}}}\left[\cos(\omega t - \theta) - \left(\frac{C^2+D^2}{C_1^2+D_1^2}\right)^{\frac{1}{2}} \cos(\omega t - \theta + \phi)\right]$$

$$+\frac{4A\omega^2a^6}{\nu\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\exp\left\{-\left(p^2+\frac{(2n-1)^2\pi^2)}{a^2}\right)\nu t\right\}\cos\{(2n-1)(2y-a)\pi/2a\}}{(2n-1)[p^2a^2+(2n-1)^2\pi^2][\nu^2\{p^2a^2+(2n-1)^2\pi^2\}^2+\omega^2a^4]} \qquad (11.0)$$

where,

$$C = \sinh(r^{\frac{1}{2}}\cos\theta/2), \qquad D = \cos(r^{\frac{1}{2}}\sin\theta/2) \qquad \dots (11.1)$$

$$C_1 = \sinh(r_1^{\frac{1}{2}}\cos\theta/2), \qquad D_1 = \cos(r_1^{\frac{1}{2}}\sin\theta/2) \qquad \dots (11.2)$$

$$r^{\frac{1}{2}} = \frac{\Omega^{1/4}}{\nu^{\frac{1}{2}}}\left(y-\frac{a}{2}\right), \qquad r_1^{\frac{1}{2}} = \frac{\Omega^{1/4}}{\nu^{\frac{1}{2}}}\frac{a}{2}, \qquad \theta = \tan^{-1}\frac{\omega}{\nu p^2} \qquad \dots (11.3)$$

$$\phi = \tan^{-1}\frac{\tanh(r_1^{\frac{1}{2}}\cos\theta/2)\tan(r_1^{\frac{1}{2}}\sin\theta/2) - \tanh(r^{\frac{1}{2}}\cos\theta/2)\tan(r^{\frac{1}{2}}\sin\theta/2)}{1+\tanh(r_1^{\frac{1}{2}}\cos\theta/2)\tan(r_1^{\frac{1}{2}}\sin\theta/2)\tanh(r^{\frac{1}{2}}\cos\theta/2)\tan(r^{\frac{1}{2}}\sin\theta/2)} \qquad \dots (11.4)$$

Thus the expression for u in (11.0) is predominantly transient in nature. An equivalent expression for u in the absence of the magnetic field is obtained by putting $H_y = 0$, *i.e.*, $p = 0$ in (11.0), so that

$$u = \frac{A}{\omega}\left[\sin\omega t - \left(\frac{k^2+L^2}{k_1^2+L_1^2}\right)^{\frac{1}{2}}\sin(\omega t - \phi_1)\right]$$

$$+\frac{4A\omega^2a^6}{\nu\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\exp\left\{-(2n-1)^2\pi^2.\frac{\nu t}{a^2}\right\}\cos\{(2n-1)(2y-a)\pi/2a\}}{(2n-1)^3\pi^2[\nu^2\pi^4(2n-1)^4+\omega^2a^4)} \qquad \dots (12.0)$$

where, $K = \sinh(r/2)^{\frac{1}{2}}, \qquad L = \cos(r/2)^{\frac{1}{2}}$ ... (12.1)

$$K_1 = \sinh(r_1/2)^{\frac{1}{2}}, \qquad L_1 = \cos(r_1/2)^{\frac{1}{2}} \quad \dots (12.2)$$

$$r^{\frac{1}{2}} = \left(\frac{\omega}{\nu}\right)^{\frac{1}{2}} (y-a/2), \quad r_1 = \left(\frac{\omega}{\nu}\right)^{\frac{1}{2}} a/2 \quad \dots (12.3)$$

$$\phi_1 = \tan^{-1} \frac{\tanh(r_1/2)^{\frac{1}{2}} \tan(r_1/2)^{\frac{1}{2}} - \tanh(r/2)^{\frac{1}{2}} \tan(r/2)^{\frac{1}{2}}}{1+\tanh(r/2)^{\frac{1}{2}} \tan(r_1/2)^{\frac{1}{2}} \tanh(r/2)^{\frac{1}{2}} \tan(r/2)^{\frac{1}{2}}}. \quad \dots (12.4)$$

The two paramoters pa and $a^2/\nu t$ appearing in (11.0) and (12.0) however are very important in so far as MHD and fluid mechanics are concerned. By (3 1) the first parameter $pa = \mu H_y a(\sigma/\nu\rho)^{\frac{1}{2}} = M$, commonly known as the Hartmann number and is very significant. In fact when M is small, viscosity of the liquid dominates over the induction drag, whereas, for large M, viscosity is unimportant relative to the magnetic viscous drag that checks any tendency to instability (Cowling 1958).

The second paramtor $a^2/\nu t$ in (12.0) known as the Reynolds number, characterizes the nature of two types of motion of a viscous fluid, laminar and turbulent. Appearance of a laminar motion in the present case requires the disappearance of the transient part in (12.0) corresponding to smaller value of Reynolds number. For larger values of this parameter the transient part does not vanish and the motion is turbulent Thus the flow of a slightly viscous fluid is characterized by large Reynolds Number with value of this parameter smaller for a laminar than for a turbulent flow. It appears therefore that there is a critical Reynolds number at which the transition from laminar to turbulent motion takes place. The upper limit of this critical number is indeterminate from the present theory but there must exist a lower limit to this number below which motion is always laminar. Experiments from Reynolds time to date suggests that transition from laminar to turbulent motion occurs nearly at the same critical Reynolds number $u_{av}\, a/\nu$ (Yuan 1969) and that lower limiting value of it is of the order 10^3

Equivalent instability conditions for magnetohydrodynamic flow do not only depend upon the Reynolds number but also on the Hartmann number which is defined as the ratio of the magnetic viscous force per unit volume to the ordinary viscous force In magnetohydrodynamics magnetic viscous force is dominant in the fluid motion and so M is large compared to unity. Naturally the lower limiting value of the critical Reynolds number is larger in this case than that in ordinary fluid mechanics.

Equation (11 0) in the present theory points out that since R is large at the limit of instability, $p^2\nu t = M^2/R$ is small for moderate value of M and can be neglected. This is quite in accordance with the investigation made by Lock

(1955). Experiments with values of M upto 130 have been carried out and the critical value of the Reynolds number is found to only 225 M.

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